Dynamic Matching and Bargaining Games: A General Approach

Stephan Lauermann   University of Michigan

Dynamic Matching and Bargaining Games

Dynamic Matching and Bargaining Games:
Provide Models of decentralized markets

Markets with decentralized features:
Labour, used cars, housing

Decentralized Market:

- Need to find trading partner
- Need to negotiate

$->$ *Frictions* of Trade

When *frictions* vanish, are decentralized markets efficient?

Motivation

- When *frictions* vanish, are decentralized markets efficient?

Motivation

- When *frictions* vanish, are decentralized markets efficient?
- Diverging Answers:

Diamond (JET, 1971)	No
Gale (JET, 1987)	Yes
DeFraja Sakovics (JPE, 2001)	Maybe
Serrano (JME, 2002)	Maybe
Moreno Wooders (JET, 2002)	Yes
Mortensen Wright (IER, 2002)	Yes
Satterthwaite Shneyerov (Econometrica, 2007)	Yes
Shneyerov (Mimeo, 2006)	Maybe
Atakan (Mimeo, 2006)	Maybe
...	...

"Maybe": Inefficiencies can persist in the limit
(e.g., depending on parameters)

Idea and Outline

Goal

- Provide an axiomatic approach to limit results of dynamic matching and bargaining games
- Use this approach to clarify divergent results in the literature

Outline

- Setup of Dynamic Matching and Bargaining Games
    - The Economy
    - The Game
- An Example
- Axiomatic approach
    - Introduce four conditions
    - Conditions are necessary and sufficient for Walrasian limit

The Quasilinear Economy

Unit Mass of Buyers and Sellers:

- Buyers: Valuation $v \in [0, 1]$, distributed acc. to $G^B(\cdot)$
- Sellers: Cost $c \in [0, 1]$, distributed acc. to $G^S(\cdot)$

$G^S(\cdot)$ and $1 - G^B(\cdot)$ are "supply" and "demand"

"Walrasian" price p^w:

$$G^S(p^w) = 1 - G^B(p^w)$$

Setup of a Dynamic Matching and Bargaining Game

Gale (1987), Satterthwaite&Shneyerov (2005)

In every period $t \in \{-\infty, \infty\}$:

1. Pool of buyers and sellers;
   Valuations v and costs c distributed acc. to $\Phi^S(\cdot)$ and $\Phi^B(\cdot)$
2. Matching into groups
3. Bargaining within groups
4. Exit:
   - All agents who traded
   - A share $\delta \in (0,1)$ of all other agents
5. Entry:
   - A mass one of buyers enter; v distributed acc. to $G^B(\cdot)$
   - A mass one of sellers enter; c distributed acc. to $G^S(\cdot)$

Setup of a Dynamic Matching and Bargaining Game

Gale (1987), Satterthwaite&Shneyerov (2005)

In every period $t \in \{-\infty, \infty\}$:

1. Pool of buyers and sellers;
   Valuations v and costs c distributed acc. to $\Phi^S(\cdot)$ and $\Phi^B(\cdot)$
2. Matching into groups
   -
3. Bargaining within groups
   -
   -
4. Exit:
   - All agents who traded
   - A share $\delta \in (0, 1)$ of all other agents
5. Entry:
   - A mass one of buyers enter; v distributed acc. to $G^B(\cdot)$
   - A mass one of sellers enter; c distributed acc. to $G^S(\cdot)$

Setup of a Dynamic Matching and Bargaining Game

An Example: price setting with asymmetric information

In every period $t \in \{-\infty, \infty\}$:

1. Pool of buyers and sellers;
   Valuations v and costs c distributed acc. to $\Phi^S(\cdot)$ and $\Phi^B(\cdot)$
2. Matching into groups
3. Bargaining within groups
4. Exit:
   - All agents who traded
   - A share $\delta \in (0, 1)$ of all other agents
5. Entry:
   - A mass one of buyers enter; v distributed acc. to $G^B(\cdot)$
   - A mass one of sellers enter; c distributed acc. to $G^S(\cdot)$

Setup of a Dynamic Matching and Bargaining Game

An Example: price setting with asymmetric information

In every period $t \in \{-\infty, \infty\}$:

1. Pool of buyers and sellers;
   Valuations v and costs c distributed acc. to $\Phi^S(\cdot)$ and $\Phi^B(\cdot)$
2. Matching into groups
   - All buyers and sellers are matched into pairs
3. Bargaining within groups
   - Seller makes an offer $p \in [0, 1]$ (without observing v)
   - Buyer accepts or rejects
4. Exit:
   - All agents who traded
   - A share $\delta \in (0, 1)$ of all other agents
5. Entry:
   - A mass one of buyers enter; v distributed acc. to $G^B(\cdot)$
   - A mass one of sellers enter; c distributed acc. to $G^S(\cdot)$

Setup of a Dynamic Matching and Bargaining Game

A Class of Games

In every period $t \in \{-\infty, \infty\}$:

1. Pool of buyers and sellers;
   Valuations v and costs c distributed acc. to $\Phi^S(\cdot)$ and $\Phi^B(\cdot)$
2. Matching into groups
3. Bargaining within groups
4. Exit:
   - All agents who traded
   - A share $\delta \in (0, 1)$ of all other agents
5. Entry:
   - A mass one of buyers enter; v distributed acc. to $G^B(\cdot)$
   - A mass one of sellers enter; c distributed acc. to $G^S(\cdot)$

Setup of a Dynamic Matching and Bargaining Game

A Class of Games

In every period $t \in \{-\infty, \infty\}$:

1. Pool of buyers and sellers;
   Valuations v and costs c distributed acc. to $\Phi^S(\cdot)$ and $\Phi^B(\cdot)$
2. Matching into groups
   - A matching function $X\left(\Phi^S, \Phi^B\right)$
3. Bargaining within groups
   - A bargaining game $\Gamma(X, V) : A \rightarrow \left[q^S, q^B, t^S, t^B\right]$
   - Speficies trading probabilities and transfers
4. Exit:
   - All agents who traded
   - A share $\delta \in (0, 1)$ of all other agents
5. Entry:
   - A mass one of buyers enter; v distributed acc. to $G^B(\cdot)$
   - A mass one of sellers enter; c distributed acc. to $G^S(\cdot)$

Translate Equilibrium into Outcomes

Idea:
For a given economy G^S, G^B and friction δ

- Derive equilibrium σ^* of $\langle \Gamma, X \rangle$
- Calculate *outcome* A induced by σ^*

Outcome $A = \left[V^S, V^B, Q^S, Q^B \right]$

- $V^S\left(c\right)$ and $V^B\left(v\right)$ are equilibrium payoffs
- $Q^S\left(c\right)$ and $Q^B\left(v\right)$ are equilibrium *lifetime trading probabilities.*

Translate Equilibrium into Outcomes

Idea:

For a given economy G^S, G^B and friction δ

- Derive equilibrium σ^* of $\langle \Gamma, X \rangle$
- Calculate *outcome* A induced by σ^*

Outcome $A = \left[V^S, V^B, Q^S, Q^B \right]$

- $V^S(c)$ and $V^B(v)$ are equilibrium payoffs
- $Q^S(c)$ and $Q^B(v)$ are equilibrium *lifetime trading probabilities.*

Dynamic Matching and Bargaining Game $\langle \Gamma, X \rangle$:

- $\left\langle \left(G^S, G^B \right); \ \delta \right\rangle \rightarrow A$

Translate Equilibrium into Outcomes

Idea:
For a given economy G^S, G^B and friction δ

- Derive equilibrium σ^* of $\langle \Gamma, X \rangle$
- Calculate *outcome* A induced by σ^*

Outcome $A = \left[V^S, V^B, Q^S, Q^B \right]$

- $V^S\left(c\right)$ and $V^B\left(v\right)$ are equilibrium payoffs
- $Q^S\left(c\right)$ and $Q^B\left(v\right)$ are equilibrium *lifetime trading probabilities.*

Dynamic Matching and Bargaining Game $\langle \Gamma, X \rangle$:

- $\left\langle \left(G^S, G^B\right);\ \delta \right\rangle \longrightarrow A$
- $\left\langle \left(G^S, G^B\right);\ \left\{\delta_k\right\}_{k=1}^{\infty} \right\rangle \longrightarrow \left\{A_k\right\}_{k=1}^{\infty}$

The Conditions

Monotonicity Condition:
Trading probabilities are increasing in valuations and decreasing in costs.

No Rent Extraction
Payoff Differences are bounded: If a seller has a cost advantage Δc over another seller, then he gets at least the same payoff and at most Δc more.

Availability
If a set of buyers trades with probability less than one, a seller who is just waiting will eventually be matched with them when δ becomes zero.

Weak pairwise efficiency
If some buyer v and seller c know they will eventually be matched with each other (or even better types) their payoffs are at least $(v - c)$ when δ becomes zero.

Monotonicity Condition

Trading probabilities Q^B, Q^S are increasing in valuations v and decreasing in costs c

Condition

Monotonicity. *A sequence of outcomes satisfies Monotonicity if the limits of the trading probabilities are monotone (whenever pointwise limits exist).*

No Rent Extraction

Payoff differences are bounded:

Condition

No Rent Extraction. *A sequence of outcomes satisfies No Rent Extraction if the slope of the limit of payoffs* V_k^B *is in* $[0,1]$ *(whenever the limit exists). In addition, if* $\lim Q_k^B(v_x) = 1$, *then*

$$\bar{V}^B(v) \geq \bar{V}^B(v_x) + (v - v_x) \text{ for all } v.$$

and similarly for sellers.

No Rent Extraction

Payoff differences are bounded:

Condition

No Rent Extraction. *A sequence of outcomes satisfies No Rent Extraction if the slope of the limit of payoffs* V_k^B *is in* $[0,1]$ *(whenever the limit exists). In addition, if* $\lim Q_k^B(v_x) = 1$, *then*

$$\bar{V}^B(v) \geq \bar{V}^B(v_x) + (v - v_x) \text{ for all } v.$$

and similarly for sellers.

Lemma

If the game $\langle \Gamma, X \rangle$ *is a game with asymmetric information, Monotonicity and No Rent Extraction hold*

Setup of a Dynamic Matching and Bargaining Game

An Example: price setting with asymmetric information

In every period $t \in \{-\infty, \infty\}$:

1. Pool of buyers and sellers;
   Valuations v and costs c distributed acc. to $\Phi^S(\cdot)$ and $\Phi^B(\cdot)$
2. Matching into groups
3. Bargaining within groups
4. Exit:
   - All agents who traded
   - A share $\delta \in (0,1)$ of all other agents
5. Entry:
   - A mass one of buyers enter; v distributed acc. to $G^B(\cdot)$
   - A mass one of sellers enter; c distributed acc. to $G^S(\cdot)$

Setup of a Dynamic Matching and Bargaining Game

An Example: price setting with asymmetric information

In every period $t \in \{-\infty, \infty\}$:

1. Pool of buyers and sellers;
   Valuations v and costs c distributed acc. to $\Phi^S(\cdot)$ and $\Phi^B(\cdot)$
2. Matching into groups
   - All buyers and sellers are matched into pairs
3. Bargaining within groups
   - Seller makes an offer $p \in [0, 1]$ (without observing v)
   - Buyer accepts or rejects
4. Exit:
   - All agents who traded
   - A share $\delta \in (0, 1)$ of all other agents
5. Entry:
   - A mass one of buyers enter; v distributed acc. to $G^B(\cdot)$
   - A mass one of sellers enter; c distributed acc. to $G^S(\cdot)$

Failure of the No Rent Extraction Condition:

Price setting with symmetric information

In every period $t \in \{-\infty, \infty\}$:

1. Pool of buyers and sellers;
   Valuations v and costs c distributed acc. to $\Phi^S(\cdot)$ and $\Phi^B(\cdot)$
2. Matching into groups
3. Bargaining within groups
4. Exit:
   - All agents who traded
   - A share $\delta \in (0,1)$ of all other agents
5. Entry:
   - A mass one of buyers enter; v distributed acc. to $G^B(\cdot)$
   - A mass one of sellers enter; c distributed acc. to $G^S(\cdot)$

Failure of the No Rent Extraction Condition:

Price setting with symmetric information

In every period $t \in \{-\infty, \infty\}$:

1. Pool of buyers and sellers;
   Valuations v and costs c distributed acc. to $\Phi^S(\cdot)$ and $\Phi^B(\cdot)$
2. Matching into groups
   - All buyers and sellers are matched into pairs
3. Bargaining within groups
   - Seller makes an offer $p \in [0, 1]$ (after observing v)
   - Buyer accepts or rejects
4. Exit:
   - All agents who traded
   - A share $\delta \in (0, 1)$ of all other agents
5. Entry:
   - A mass one of buyers enter; v distributed acc. to $G^B(\cdot)$
   - A mass one of sellers enter; c distributed acc. to $G^S(\cdot)$

Availability

If a set of buyers trades with probability less than one, a seller who is just waiting will eventually be matched with them when $\delta \longrightarrow 0$.

Condition

Availability. *A sequence* $\{A_k\}_{k=1}^{\infty}$ *satisfies Availability relative to some function* L^B *if, whenever* $\limsup Q_k^B(v) < 1$*, for all* $v < v_x$ *then* $\lim L^B(v, \delta_k, A_k) = 1$ *for all* $v < v_x$ *(and similarly for seller).*

$L^B(v_x, \delta_k, A_k)$ is the probability that a seller who is just passively waiting in the pool will be matched at least once with a buyer of type $v \geq v_x$.

Availability

If a set of buyers trades with probability less than one, a seller who is just waiting will eventually be matched with them when $\delta \to 0$.

Condition

Availability. *A sequence* $\{A_k\}_{k=1}^{\infty}$ *satisfies Availability relative to some function* L^B *if, whenever* $\limsup Q_k^B(v) < 1$, *for all* $v < v_x$ *then* $\lim L^B(v, \delta_k, A_k) = 1$ *for all* $v < v_x$ *(and similarly for seller).*

Availability

If a set of buyers trades with probability less than one, a seller who is just waiting will eventually be matched with them when $\delta \to 0$.

Condition

Availability. *A sequence* $\{A_k\}_{k=1}^{\infty}$ *satisfies Availability relative to some function* L^B *if, whenever* $\limsup Q_k^B(v) < 1$, *for all* $v < v_x$ *then* $\lim L^B(v, \delta_k, A_k) = 1$ *for all* $v < v_x$ *(and similarly for seller).*

Suppose some set of types makes up a strictly positive share of the pool. A matching function X is *not vanishing*, if the probability to be matched with a type from this set is strictly positive and if it is continuous in the shares.

Lemma

If the matching function X *is not vanishing, then availability holds.*

Weak Pairwise Efficiency

If some buyer v and seller c know they will eventually be matched with each other (or even better types) then their payoffs are at least $(v - c)$.

Condition

Weak pairwise efficiency. *A sequence $\{A_k\}_{k=1}^{\infty}$ satisfies Weak Pairwise Efficiency if* $\lim L_k^S(c_x) = 1$ *and* $\lim L_k^B(v_x) = 1$ *implies*

$$\liminf \left(V_k^S(c_x) + V_k^B(v_x) \right) \geq v_x - c_x .$$

Failure of Weak Efficiency

Simultaneous Double Auction (Serrano, JME, 2002)

In every period $t \in \{-\infty, \infty\}$:

1. Pool of buyers and sellers;
   Valuations v and costs c distributed acc. to $\Phi^S(\cdot)$ and $\Phi^B(\cdot)$
2. Matching into groups
3. Bargaining within groups
4. Exit:
   - All agents who traded
   - A share $\delta \in (0,1)$ of all other agents
5. Entry:
   - A mass one of buyers enter; v distributed acc. to $G^B(\cdot)$
   - A mass one of sellers enter; c distributed acc. to $G^S(\cdot)$

Failure of Weak Efficiency

Simultaneous Double Auction (Serrano, JME, 2002)

In every period $t \in \{-\infty, \infty\}$:

1. Pool of buyers and sellers;
   Valuations v and costs c distributed acc. to $\Phi^S(\cdot)$ and $\Phi^B(\cdot)$
2. Matching into groups
   - All buyers and sellers are matched into pairs
3. Bargaining within groups
   - Seller and buyer simultaneously announce bids p and r
   - Trade happens at p if $p \leq r$
4. Exit:
   - All agents who traded
   - A share $\delta \in (0, 1)$ of all other agents
5. Entry:
   - A mass one of buyers enter; v distributed acc. to $G^B(\cdot)$
   - A mass one of sellers enter; c distributed acc. to $G^S(\cdot)$

Summary

Monotonicity Condition:
Trading probabilities are increasing in valuations and decreasing in costs.

No Rent Extraction
Payoff Differences are bounded: If a seller has a cost advantage Δc over another seller, then he gets at least the same payoff and at most Δc more.

Availability
If a set of buyers trades with probability less than one, a seller who is just waiting will eventually be matched with them when δ becomes zero.

Weak pairwise efficiency
If some buyer v and seller c know they will eventually be matched with each other (or even better types) their payoffs are at least $(v - c)$ when δ becomes zero.

Main Result

Theorem

Suppose some sequence $\{A_k\}_{k=1}^{\infty}$ *satisfies mass balance and has uniformly bounded variation. Then the limiting outcome becomes Walrasian, i.e.,* $\lim A_k = A^W$*, if and only if* $\{A_k\}_{k=1}^{\infty}$ *satisfies Monotonicity, No Rent Extraction, Availability, and Weak Pairwise Efficiency.*

Main Result

Corollary

Suppose outcomes $\{A_k\}$ *are generated by a game from the class* $\langle \Gamma, X \rangle$*. Suppose that the bargaining game is a game with asymmetric information and suppose it is not too inefficient. Suppose in addition that the matching technology* X *is not vanishing. Then the limiting outcome is Walrasian*

$$\lim_{k \to \infty} A_k = A^W .$$

Sufficiency Condition: Pairwise Efficiency

Lemma

Suppose some sequence $\{A_k\}_{k=1}^{\infty}$ *is of uniformly bounded variation and suppose that for all* v *and* c,

$$\liminf_{k\to\infty}\left[V_k^S(c)+V_k^B(v)\right]\geq v-c \qquad \textit{then} \qquad \lim_{k\to\infty} S(A_k)=S^*.$$

$S(A_k)$: The realized trading surplus of an outcome A_k

Summary of the Proof of Efficiency

Monotonicity $\Rightarrow$ cutoff types c_x and v_x, $c_x \equiv \inf\left\{c, 1 \middle| \bar{Q}^S(c) < 1\right\}$

Availability and Weak efficiency:

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq (v_x - c_x)$$

No Rent Extraction:

$$\begin{array}{rrcll} & \bar{V}^S(c) & \geq & \bar{V}^S(c_x) + (c_x - c) & \text{for all } c, \\ \text{and} & \bar{V}^B(v) & \geq & \bar{V}^B(v_x) + (v - v_x) & \text{for all } v. \end{array}$$

Together:

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c$$

Payoffs are pairwise efficient for **all** types $\Rightarrow$ Efficiency

Monotonicity: Define Cutoff Types

Observation 1: There exists a convergent subsequence $\{A_{k'}\}$ with pointwise limit $\left[\bar{V}^S, \bar{V}^B, \bar{Q}^S, \bar{Q}^B\right]$. All component of this limiting outcome are monotone.

By the monotonicity condition and by the no rent extraction condition

Define cutoff types

$$c_x \equiv \inf\left\{c, 1 | \bar{Q}^S(c) < 1\right\} \quad \text{and} \quad v_x \equiv \sup\left\{v, 0 | \bar{Q}^B(v) < 1\right\}.$$

Monotonicity: Define Cutoff Types

Observation 1: There exists a convergent subsequence $\{A_{k'}\}$ with pointwise limit $\left[\bar{V}^S, \bar{V}^B, \bar{Q}^S, \bar{Q}^B\right]$. All component of this limiting outcome are monotone.

By the monotonicity condition and by the no rent extraction condition

Define cutoff types

$$c_x \equiv \inf \left\{ c, 1 | \bar{Q}^S(c) < 1 \right\} \quad \text{and} \quad v_x \equiv \sup \left\{ v, 0 | \bar{Q}^B(v) < 1 \right\}.$$

Case: $0 < c_x \leq v_x < 1$ (*The other cases follow similarly*)

Availability and Weak efficiency

Observation 2: $\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq (v_x - c_x)$

By definition $\bar{Q}^S(c_x + \varepsilon) < 1$ *and* $\bar{Q}^B(v_x - \varepsilon) < 1$ *for* ε *small enough*

These types are available:

$$\bar{L}^S(c_x + \varepsilon) = \bar{L}^B(v_x - \varepsilon) = 1$$

Their joint payoff is pairwise efficient

$$\bar{V}^S(c_x + \varepsilon) + \bar{V}^B(v_x - \varepsilon) \geq (v_x - c_x) - 2\varepsilon$$

From continuity of $\bar{V}^S(\cdot)$ *and* $\bar{V}^B(\cdot)$ *(implied by no-rent extraction):*

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq (v_x - c_x)$$

No Rent Extraction

Observation 3: Payoffs are bounded from below:

$$\begin{array}{lrcll} & \bar{V}^S(c) & \geq & \bar{V}^S(c_x) + (c_x - c) & \text{for all } c, \\ \text{and} & \bar{V}^B(v) & \geq & \bar{V}^B(v_x) + (v - v_x) & \text{for all } v. \end{array}$$

From definition: $\bar{Q}^S(c_x - \varepsilon) = 1$

From the no rent extraction condition:

$$\bar{V}^S(c) \geq \bar{V}^S(c_x - \varepsilon) + (c_x - c) - \varepsilon$$

and by continuity of $\bar{V}^S(\cdot)$*:*

$$\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c)$$

Putting Things Together:

Observation 5: $\bar{V}^S(c) + \bar{V}^B(v) \geq v - c \qquad$ for all v and c

By adding up the inequalities of observation 3:

$$\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c) \quad \text{and} \quad \bar{V}^B(v) \geq \bar{V}^B(v_x) + (v - v_x)$$

$$\Rightarrow \bar{V}^S(c) + \bar{V}^B(v) \geq v - c + \underbrace{\bar{V}^S(c_x) + \bar{V}^B(v_x) - (v_x - c_x)}$$

Putting Things Together:

Observation 5: $\bar{V}^S(c) + \bar{V}^B(v) \geq v - c$ for all v and c

By adding up the inequalities of observation 3:

$$\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c) \quad \text{and} \quad \bar{V}^B(v) \geq \bar{V}^B(v_x) + (v - v_x)$$

$$\Rightarrow \bar{V}^S(c) + \bar{V}^B(v) \geq v - c + \underbrace{\bar{V}^S(c_x) + \bar{V}^B(v_x) - (v_x - c_x)}_{\geq 0 \text{ by observation 4}}$$

–> *So the limiting outcome is pairwise efficient for all types*

Taking care of the remaining cases completes the proof.

Summary of the Proof

Monotonicity ensures existence of cutoff types c_x and v_x

Availability and Weak efficiency:

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq (v_x - c_x)$$

No Rent Extraction:

$$\begin{array}{lrcll} & \bar{V}^S(c) & \geq & \bar{V}^S(c_x) + (c_x - c) & \text{for all } c, \\ \text{and} & \bar{V}^B(v) & \geq & \bar{V}^B(v_x) + (v - v_x) & \text{for all } v. \end{array}$$

Together:

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c$$

Payoffs are pairwise efficient for all types

Conclusion

- Axiomatic Approach to Limit Results in Dynamic Matching and Bargaining Games
- Four conditions are necessary and sufficient for efficiency: Monotonicity, No Rent Extraction, Availability, and Weak Efficiency
- Conditions hold in many specifications of DMBG
  - Highlights "Common Cause" across models
- Failure of efficiency relates to failure of specific conditions

Open Questions

Open Questions

- More complex economies?
- Behavioral Strategies instead of Bayesian Nash?
- Limit results in other classes of games?
  - $\frac{1}{\delta}$ as the number of Bidders in Double Auctions
- Aggregate uncertainty about p^w?
  - Wolinsky (1990): Efficiency impossible with common values

Proof:

Monotonicity: Define Cutoff Types

Observation 1: There exists a convergent subsequence $\{A_{k'}\}$ with pointwise limit $\left[\bar{V}^S, \bar{V}^B, \bar{Q}^S, \bar{Q}^B\right]$

(By Helly's selection principle)

Observation 2: The **monotonicity condition** allows us to define cutoff types

$$c_x \equiv \inf\left\{c, 1 \middle| \bar{Q}^S\left(c\right) < 1\right\} \quad \text{and} \quad v_x \equiv \sup\left\{v, 0 \middle| \bar{Q}^B\left(v\right) < 1\right\}.$$

Continue working with this subsequence and its limit

Assumption: $0 < c_x \leq v_x < 1$ *(The other cases follow similarly)*

Availability and Weak efficiency

Observation 3: $\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq (v_x - c_x)$

By definition $\bar{Q}^S(c_x + \varepsilon) < 1$ *and* $\bar{Q}^B(v_x - \varepsilon) < 1$ *for* ε *small enough*

These types are available:

$$\bar{L}^S(c_x + \varepsilon) = \bar{L}^B(v_x - \varepsilon) = 1$$

Their joint payoff is pairwise efficient

$$\bar{V}^S(c_x + \varepsilon) + \bar{V}^B(v_x - \varepsilon) \geq (v_x - c_x) - 2\varepsilon$$

From continuity of $\bar{V}^S(\cdot)$ *and* $\bar{V}^B(\cdot)$ *(implied by no-rent extraction):*

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq (v_x - c_x)$$

No Rent Extraction

Observation 4: Payoffs are bounded from below:

$$\begin{array}{lrcll} & \bar{V}^S(c) & \geq & \bar{V}^S(c_x) + (c_x - c) & \text{for all } c, \\ \text{and} & \bar{V}^B(v) & \geq & \bar{V}^B(v_x) + (v - v_x) & \text{for all } v. \end{array}$$

From definition: $\bar{Q}^S(c_x - \varepsilon) = 1$

From the no rent extraction condition:

$$\bar{V}^S(c) \geq \bar{V}^S(c_x - \varepsilon) + (c_x - c) - \varepsilon$$

and by continuity of $\bar{V}^S(\cdot)$*:*

$$\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c)$$

Putting Things Together:

Observation 5: $\bar{V}^S(c) + \bar{V}^B(v) \geq v - c \qquad$ for all v and c

By adding up the inequalities of observation 3:

$$\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c) \quad \text{and} \quad \bar{V}^B(v) \geq \bar{V}^B(v_x) + (v - v_x)$$

$$\Rightarrow \bar{V}^S(c) + \bar{V}^B(v) \geq v - c + \underbrace{\bar{V}^S(c_x) + \bar{V}^B(v_x) - (v_x - c_x)}$$

Putting Things Together:

Observation 5: $\bar{V}^S(c) + \bar{V}^B(v) \geq v - c$ for all v and c

By adding up the inequalities of observation 3:

$$\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c) \quad \text{and} \quad \bar{V}^B(v) \geq \bar{V}^B(v_x) + (v - v_x)$$

$$\Rightarrow \bar{V}^S(c) + \bar{V}^B(v) \geq v - c + \underbrace{\bar{V}^S(c_x) + \bar{V}^B(v_x) - (v_x - c_x)}_{\geq 0 \text{ by observation 4}}$$

–> *So the limiting outcome is pairwise efficient for all types*

Putting Things Together:

Observation 5: $\bar{V}^S(c) + \bar{V}^B(v) \geq v - c \qquad$ for all v and c

By adding up the inequalities of observation 3:

$$\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c) \quad \text{and} \quad \bar{V}^B(v) \geq \bar{V}^B(v_x) + (v - v_x)$$

$$\Rightarrow \bar{V}^S(c) + \bar{V}^B(v) \geq v - c + \underbrace{\bar{V}^S(c_x) + \bar{V}^B(v_x) - (v_x - c_x)}_{\geq 0 \text{ by observation 4}}$$

–> *So the limiting outcome is pairwise efficient for all types*

Observation 6: *For the convergent subsequence* $\{A_{k'}\}$

$$\lim_{k' \to \infty} S(A_{k'}) = S^*$$

Taking care of the remaining cases completes the proof of convergence to efficiency. Link: Back

Walrasian Outcome and Necessity

Walrasian Outcome A^W
Trading probabilities Q^S,Q^B are one if and only if $c < p^w$ and $v > p^w$
Payoffs are $\max\{p^w - c, 0\}$ and $\max\{v - p^w, 0\}$

Efficient implies that trading probabilities $\bar{Q}^S$,$\bar{Q}^B$ are Walrasian

Walrasian Outcome and Necessity

Walrasian Outcome A^W
Trading probabilities Q^S,Q^B are one if and only if $c < p^w$ and $v > p^w$
Payoffs are $\max\left\{p^w - c, 0\right\}$ and $\max\left\{v - p^w, 0\right\}$

Efficient implies that trading probabilities $\bar{Q}^S$,$\bar{Q}^B$ are Walrasian

No rent extraction implies payoffs are *at least* Walrasian
$V^S\left(c\right) \geq p^w - c$ and $V^B\left(v\right) \geq v - p^w$.

Payoffs cannot be larger than efficient payoffs, therefore payoffs *are exactly* Walrasian

Walrasian Outcome and Necessity

Walrasian Outcome A^W
Trading probabilities Q^S,Q^B are one if and only if $c < p^w$ and $v > p^w$
Payoffs are $\max\{p^w - c, 0\}$ and $\max\{v - p^w, 0\}$

Walrasian Outcome and Necessity

Walrasian Outcome A^W
Trading probabilities Q^S,Q^B are one if and only if $c < p^w$ and $v > p^w$
Payoffs are $\max\{p^w - c, 0\}$ and $\max\{v - p^w, 0\}$

If $A_k \rightarrow A^W$ then

- Trading probabilities are monotone
- Payoffs satisfy no rent extraction
- Availability and Weak Efficiency holds:

$\bar{V}^B(v) + \bar{V}^S(c) \geq (v - p^w) + (p^w - c)$ for all v, c Link: Back